THE DEPARTMENTS OF MATHEMATICS, AND THEIR MUTUAL RELATIONS.

By George H. Howison.

I. — HISTORICAL.

The classification of the Mathematical Sciences is a comparatively recent form of exercise for the human intelligence. Indeed, to the ordinary reader, the multitudinous variety of the branches seems to preclude the conception of a harmonious whole. Merely to read the list, following as far as possible the order of study, — Arithmetic, Algebra; Geometry, Trigonometry, Mensuration, Plane Surveying, Navigation; Conic Sections and Analytic Geometry; the Calculus of Finite Differences, the Differential and Integral Calculus, and the Calculus of Variations; Mechanics; Astronomy, Theoretical and Practical; Geodesy, Topographical Surveying, Nautical Astronomy; Engineering, Civil, Military, Naval, and Mechanical; Mining and Architectural Calculations, and Stereotomy; Physico-Mathematics, including Optics, Thermodynamics, Acoustics, and Molecular Mechanics—is not the imagination at once confused by the interminable array of topics? The differences, rather than the resemblances, of the various members of the series strike the attention. We seem to be caught away in an indistinguishable whirl,—the whole appearing as " a mighty maze, and *all* without a plan." Or, if to the thoughtful student there comes to appear a certain coherence and development in the elementary branches of Arithmetic, Algebra, and Geometry, he finds the continuity of his conceptions suddenly broken up when he reaches the province of Analytic Geometry. The interruption only extends itself when he enters the misty region of the Infinitesimal Calculus, and finally leaves him drifting about amid the tantalizing shadows of methods that seem wholly arbitrary. And yet our list, long as it is, gives no hint of how Geometry breaks up and shades away into Plane, and Solid, and Spherical, with episodes of Planimetry and Stereometry, of Stereography and Descriptive Geometry. We have not mentioned

the so-called New Geometry, with its Transversals, its An-harmonic Ratio and Harmonic Conjugates, its Reciprocal Polars and Centres of Harmonic Means; nor the Modern Algebra, with its armament of Discriminants, Determinants, Invariants, and Co-variants; nor yet the Quaternion, just risen into view, with its Fourfold of Impossibles capable of representing every Real, and bringing in a train of Double, Triple, Sextuple, and other Multiple Algebras.

To this manifold diversity, as a source of difficulty, must be added the fact, that the system of mathematics has only unfolded itself by the slow progress of ages, and never assumed its normal proportions till toward the close of the eighteenth century. In fact, its subtlest principles never came to consciousness until the invention of the Higher Calculus in the latter part of the seventeenth century, and another century elapsed before the principles thus revealed had taken consistency and thoroughly reorganized the science. Meanwhile, the great mathematicians were so much absorbed in the exciting business of perfecting the Calculus of Leibnitz and Newton, and in the wonderful physical discoveries to which it led at once, that they had neither time nor inclination for those less fruitful meditations which the philosophy of their science demands.

The result has been, that, although every member of an enlightened community is now familiar with the details of at least one branch of mathematics; and although the elementary study of all its branches known at the time, has in all civilized ages been counted essential to liberal education; still, the co-ordination of its various parts·has, until recently, received no proper attention, and may even now be regarded as a matter of controversy. If Berkeley, D'Alembert, Carnot, and Lagrange, had succeeded in reaching the fundamental principles of some of the branches, or in re-establishing a *principle* of continuity which the Calculus of Leibnitz and Newton had seemed to interrupt, the actual co-ordination itself — the comprehension of mathematics as an organized whole — still awaited realization at the beginning of the present century. Comte, the founder of the Positive Philosophy, opens in 1829 his lectures on the *Philosophy of Mathematics* with the following notable statement:*

* *Cours de Philosophie Positive,* tome 1, p. 117.

"Although the science of mathematics is the most ancient and the most perfect of all sciences, the general conception which we should form of it is not yet clearly determined. The definition of the science, and its principal divisions, have hitherto remained vague and uncertain. The plural name by which it is usually designated would of itself be sufficient to indicate the want of unity in its philosophic character, as it is commonly conceived."

And so well founded does John Stuart Mill consider this sweeping assertion, that he declares* that Comte "may truly be said to have *created* the philosophy of the higher mathematics," and that his speculations on this subject "are among the most valuable gifts for which philosophy is indebted to that eminent thinker."

It is to be doubted whether the results of Comte's labors will justify this exalted opinion; but the bare fact of its expression by Mill is sufficient to indicate that the field of mathematical philosophy had been but little cultivated before Comte's day. It is unquestionable that Comte was deeply indebted to some of his immediate predecessors — to Carnot, for instance, and, above all, to Lagrange — but this he has himself declared, with enthusiasm;† and, when all the deductions on this score have been made, he must still be credited with the only serious attempt to set forth comprehensively, in a united system, the Idea of mathematics as a whole, and to carry out in complete detail its logical development into its several branches. To be sure, in 1755, Montucla conceived and partly executed the plan of writing a history of all mathematical science,—a work which he revised and began to republish in 1798, and which, upon his death in the midst of his labors, his friend Lalande brought to completion three years later; but this work, vast in design and compre-

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* *System of Logic*, New York, 1846, p. 369. Sixth edition, London, 1865, v. II., p. 153.

† *Cours de Philosophie Positive*, t. 1, p. 118: — "...... les derniers perfectionnements capitaux éprouvés par la science mathématique ont directement préparé cette importante opération philosophique, en imprimant à principales parties un caractère d'unité qui n'existait pas auparavant; *tel est éminemment et hors de toute comparaison l'esprit des travaux de l'immortel auteur de la 'Théorie des Fonctions' et de la 'Mécanique analytique.'* " Again, p. 244: " .... Carnot présenta enfin *la véritable explication logique directe de la méthode de Leïbnitz*.........Carnot a rendu ainsi à la science *un service essentiel, et dont l'importance me semble n'être pas encore suffisamment appréciée.*"

hensive in scope to a degree that will keep it the storehouse for generations of investigators yet to come, falls far short of coördinating the parts of the science,* and labors heavily with its enormous mass of imperfectly digested material. Prompted by his sense of these defects in the work, the mathematician Bossut, while Montucla's revision was coming through the press, published his *Histoire générale des Mathématiques*, a book characterized by thoroughness, brevity, and a clear arrangement based on an attempt to define and classify the science and its branches; but he advanced no farther than to divide mathematics, by the arbitrary rule already in vogue, into *Pure* and *Mixed :* referring to the former class Arithmetic, Geometry, Analysis, and Analytic Geometry; and to the latter, Mechanics, Hydrodynamics, Astronomy, Optics, and Acoustics. So that, while Montucla and Bossut both contributed toward *presenting* the total field of mathematics at a single view, neither of them can be said to have brought it under the control of one all-pervading Thought.

Nor do the labors of Hegel interfere seriously with the peculiar claims of Comte. For, though in 1812 Hegel devoted the whole second chapter of his celebrated *Logik* to the dialectic of Quantity, aiming to show up the transition from Quality to Quantity in the Pure Thinking, and to trace the very Ori-

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* Lest I be thought unjust to the patient and thorough Montucla, I here subjoin his attempt at classification:—

I. *Pure, or Abstract Mathematics :*
    1. Arithmetic = Science of Number;
    2. Geometry = Science of Figured Extension;
        a. Elementary, extending through the Circle,
        b. Transcendental; including the other Curves,
            α. Finite.
            β. Infinitesimal.
    3. Algebra = *the Mediation of Arith. and Geom., and including both* [strangely defined, again, as " Arith. by Signs." and as " Science of *any* Relations—of Magnitude in general"] and subdivided as
        a. Ordinary, dealing with Finites, Solution of Equations, and Theory of Curves.
        b. Infinitesimal = the Diff. and Integ. Calculus.
II. *Mixed, or Physico-Mathematics :*
    1. Mechanics,
    2. Astronomy,
    3. Acoustics,
    4. Optics, etc.

gin, or First Rise, of the conception of Quantity in the process of Intelligence; though he offered an absolute definition of Quantity in its total comprehension, and endeavored to unfold its relations to Space, Time, Motion, and Number; though he entered upon a detailed critique of the Mathematical Infinite, and believed that he had exposed the unphilosophical character of the Infinitesimal Calculus as expounded either by Newton or Leibnitz: yet his treatment of mathematics is, upon the whole, fragmentary; thoroughgoing classification is nowhere attempted in it; and his criticism (even granting that its fundamental definitions are correct, which few perhaps will concede) is at best but germinal, and not applied in detail to subordinate the science.*

So far, then, as concerns the actual grasping together of all the branches under one superintending conception, and attempting to work out their logical rise from it with something like a mastery of their details, it seems to me that Comte's claim to prominence cannot be successfully challenged. It must be remembered, however, that this claim rests on his comprehensiveness of grasp alone, some of the most important principles of his classification having been provided to his hand, as already noticed, by Lagrange mainly, and, in a lesser degree, by Berkeley† and Carnot.‡ If, then, the system which he unfolds is in some points marked by a far-and-wide-reaching insight, and in others is still open to serious criticism, the praise and dispraise should be distributed impartially among its several authors.

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* In this comparison between Comte and Hegel, let me be distinctly understood to refer to *comprehensiveness of formal method* alone: Hegel's labors occupy a different field from Comte's, and an altogether higher—as much higher as Essence is than form.

The only other writers who, so far as I am aware, have entered the field of *general* mathematical philosophy, are J. S. Mill and Professor Bledsoe of the University of Virginia. The latter published in 1868 a *Philosophy of Mathematics*, which however makes no claim to exhaustiveness, dealing mainly with Geometry and the Infinitesimal Method. The author treats Comte, and Mill's opinion of him, with undisguised contempt, asserting that he " has added not a single idea to those of his predecessors, except a few false ones of his own."

Mill's *Logic* and his *Examination of Sir W. Hamilton's Philosophy* both contain chapters giving general views of mathematics, in the Comtian spirit indeed, but under new and interesting aspects.

† Berkeley's *Works*, v. II., p. 422.

‡ Carnot, *Réflexions sur la Métaphysique du Calcul Infinitésimal.*

The very fact, however, that Comte's system thus reposes, in a certain sense, on the highest combination of mathematical and metaphysical genius that preceded him, renders it proper that any later attempt in the same field should start from a correct estimate of his work. I therefore now proceed to the details of his scheme.

Comte sets out with a critique and estimate of the current definition of mathematics — a definition that is current still. *Mathematics is the science of magnitudes,* he quotes; or, rather, *the science which has for its object the measurement of magnitudes.* Of this definition, which to the general reader, or writer, even at this day is eminently plausible, simple, and lucid, he has not a high opinion. It is " vague and meaningless"; it " has singular need to acquire more precision and depth"; it is a " rude outline"—a " scholastic glimpse." His objection to it seems fatal, however; evidently, it does not raise mathematics above the mechanical art of laying off an assumed unit on the magnitude which we wish to measure; it is silent as to any *indirect* measurement of magnitudes, by deriving their measures from the known or readily determinable measures of others with which they are connected by some known law. And yet measurements of this latter class are the staple of mathematics as it exists,—are the only ground on which the title of *science* (or methodized knowledge) can be claimed for it,—are, in Astronomy, Geodesy, Thermodynamics, Celestial Mechanics, its chief glories. How but by indirect calculation can we ever measure the distances of the planets, or prophesy the existence, place, and orbit, of one before unknown? How otherwise can we measure an arc of the meridian; or determine the mechanical equivalent of heat; or demonstrate the law that the planetary periods vary in the sesquiplicate ratio of the mean distances from the sun?

Conceding, however, that the " scholastic glimpse " is " at bottom just," in virtue of mentioning measurement of *some* kind, Comte works his way up from it by means of such considerations as have just been hinted, and arrives at what he terms a " definition worthy of the importance, the extent, and the difficulty of the science." It is as follows : — " We have now come to define the science of mathematics with exacti-

tude, by assigning as its object the *indirect* measurement of magnitudes, and saying that we constantly propose in it *to determine magnitudes one from another, according to the fixed relations which exist among them.*"*

From this definition, which, compared with the former, surely seems vital and full of meaning, Comte sets forth on the march of classification. He finds, as it implies, that every actual mathematical problem in the natural world involves two great steps: the first, the determination of the "fixed relations" among the magnitudes of natural phenomena; the second, the representation of these relations in an equation, and the final determination of the measurement sought by the numerical solution of that equation. He thus provides for two grand divisions, into one or the other of which all the mathematical sciences must fall : to the first, inasmuch as it deals (as he thinks) with the relations of natural phenomena, and seeks the *laws* (or, as he deems them, the *generalized facts*) of their quantification, he gives the name of *Concrete Mathematics ;* to the second, whose business is solely to note and reduce the forms of equations, and to combine numbers, by modes of operation determined by thought alone, and therefore independent of the merely phenomenal world, he gives the name of *Abstract Mathematics.* Of the subdivisions of this Concrete branch, he soon disposes : they are, for the time being, *two,*—Rational Mechanics and Geometry, the latter separating again into the ordinary Pure Geometry (which he calls Synthetic or Special), and Algebraic Geometry (which he names Analytic, or General). He is not sure, however, but he ought to include *Thermology*, or the laws of the quantification of Heat, as a *third* principal subdivision of Concrete Mathemathics, co-ordinate with Rational Mechanics and Geometry. This, because the investigations of Baron Fourier had just then resulted in the establishment of direct equations of heat,—algebraic expressions of the quantitative laws of heat itself, independent of any modified application of me-

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* Nous sommes donc parvenu maintenant à définir avec exactitude la science mathématique, en lui assignant pour but, la mesure *indirecte* des grandeurs, et disant qu'on s'y propose constamment de *déterminer les grandeurs les unes par les autres, d'après les relations précises qui existent entr' elles.*"—*Cours de Philosophie Positive,* t. 1, p. 129.

chanics. Thus, he evidently regarded the real number of his subdivisions of Concrete Mathematics as liable to increase at any time by the fortunate rise of physical discoveries; in fact, their number should be potentially without limit, or have at all events no other limit than man's capacity to generalize the phenomena of the physical world. And here we must note, that, in ranking Geometry and Rational Mechanics with a science so clearly empirical as Thermodynamics, Comte proceeded deliberately: it being his conviction that all geometric properties, as well as the laws of motion, were mere generalizations from our experience in the natural world. He did not even except the so-called mathematical axioms from this sweeping opinion—an opinion, by the way, in which he. enjoys the entire concurrence of John Stuart Mill.

Having thus closed the discussion of the Concrete Mathematics, Comte proceeds to unfold the subdivisions of the Abstract, whose business, as we have already seen, is actually to *derive* the measurements of required magnitudes, in accordance with the laws imposed by the Concrete; thus he finds that, in this province, every measurement, or *value*, is a *derived* value—is obtained from other *given* values by a series of definite operations, inseparable from our experience of the nature of numbers, and whose combinations vary to express the various *laws* of derivation which the nature of geometric, mechanical, or thermological phenomena may require. In a word, in his second province every required value is, as the mathematicians say, a *function* of some given value or values: it is derived from them by some uniform rule, and must vary whenever they vary. He therefore re-names his second province by the title of Analysis, or the *Calculus of Functions*, and advances to the subdivision of this calculus.

And here his doctrine that all the laws which express themselves in functions are generalized from experience, and that the equations expressing those functional laws must therefore become more and more complex as the scale of the phenomena investigated rises higher and higher in complexity, requires him to recognize two general cases: the first, where the quantifications of phenomena are sufficiently simple to

be put directly into equations; the second, where these are
so complex that we cannot reach their own relations, but are
obliged to call in the aid of certain relations among the ele-
mentary constituents of the phenomena, known in mathemat-
ics as the *differentials* of original quantities: between these
elements, whose relations we can express, and which are
connected by uniform laws of derivation with their primaries,
the principal phenomena, we then establish equations, and
apply to them a calculus which enables us to pass back and
forth from them to their primary phenomena and recipro-
cally. Thus, his general Calculus breaks up into a *Calculus
of Direct Functions*, which he calls Ordinary Analysis; and
a *Calculus of Indirect Functions*, which he names Transcen-
dental Analysis.

The Calculus of Direct Functions involves two steps: we
have to determine general laws of transformation, or general
rules for the solution of equations; and we have to apply
these to the specific numerical equation of the problem in
hand, or determine the numerical value of its roots. So Comte
passes immediately to the familiar branches of *Algebra* and
Arithmetic. Of the singular confusion in his treatment of
these, I shall speak hereafter.

The Calculus of Indirect Functions, in general, also involves
two steps, or rather phases of procedure: we may have to
pass from the primary phenomenon to the " differential " of
it; or, backward, from a given " differential " to the primary
phenomenon termed its " integral." And these two phases
Comte finds so interwoven in many problems, that he groups
them together as one, and waits for their counterpoise in his
scheme until he discusses that special case (as he calls it) in
the Integral Calculus, where we have to pass back from a
differential given under such conditions that its integral is
unknown, and is supposed to be itself gradually *changing,*
in accordance with some mathematical law. Thus he finally
subdivides his Transcendental Calculus into the *Differential
and Integral Calculus* and the *Calculus of Variations*.

His scheme may be recapitulated as follows, inverting the
order of the foregoing outline, as he himself does in the detail
of his lectures:

I. ABSTRACT MATHEMATICS = *Analysis,* or *the Calculus.*
  1. ORDINARY ANALYSIS = the Calculus of Direct Functions.
    a. Algebra = the Calculus of Functions *per se.*
    b. Arithmetic = the Calculus of Values.
  2. TRANSCENDENTAL ANALYSIS = the Calculus of Indirect Functions.
    a. Diff. and Integ. Calculus. ⎱
    b. Calculus of Variations. ⎰ [These lack a coördinating Definition.]

II. CONCRETE MATHEMATICS = Investigation of *Calculable Laws of Phenomena.*
  1. [THERMOLOGY = Math. Laws of Heat.]
  2. RATIONAL MECHANICS = Math. Laws of Physical Motion.
  3. GEOMETRY = Math. Laws of Physical Figures.
    a. Synthetic, or Special { Graphic.........Algebraic.
                            Descriptive.........Trigonometry.
    b. Analytic, or General { of Two Dimensions.
                            of Three Dimensions.

II.—CRITICAL.

The strong points of the foregoing scheme are obvious:—it sets out with a definition of the science which commends itself at once, to all who are practically conversant with mathematics, as a comprehensive and striking description of what is actually done there; its definitions of the subordinate departments have the quality, essential to science, of describing their subjects by their object-matter; and, in dividing the science into two great realms, the one occupied with the investigation and proof of those laws which govern the derivation of functions, and the other with the actual derivation (or, as the technical phrase is, the *computation*) of the functions, it is not only again in wide accord with the evident facts, but seems at once to indicate, by its luminous and penetrating principle, the pathway through any remaining intricacies of the subject. Doubtless in mathematics we *are* invariably occupied either in establishing laws which connect the value of one quantity more or less immediately with that of another, or else in actually computing such a derived value in accordance with its connecting law. Thus, in Mechanics, we establish numerical laws of motion or force—laws by which the total motion may be derived from the velocity and time, or the acquired velocity from the time and rate of acceleration, or the force from the resistance overcome; in Geometry, we establish numerical laws of form—laws by which we can more

-or less directly find the dimensions of one part of a figure from others that are given ; while, as results of these two sciences, we form equations expressing the adjustment of such laws, whether mechanical or geometric, to the conditions of any given problem, and then proceed to transform them by algebraic computation, until at last we bring the value of the required quantity within the direct domain of Arithmetic, -computing it by some one of the elementary rules. And when this broad distinction between *creating* and *computing* functions is once made, it seems as if our way were tolerably -clear to the correct subdivision of the corresponding departments.

But, if we thus recognize the correctness of Comte's scheme in the leading outlines, it seems to me that we must still decide that it is open to serious criticism in many of its most important details. Even to his definition of mathematics, exact as is its descriptive character, the careful metaphysician will have to object that it lacks, after all, the thoroughly scientific quality, because it fails entirely to connect its system of indirect measurement with the essential nature and properties of quantity in general. There being an elemental conception of intelligence called Quantity, and mathematics having by universal consent some very important relation or other to that conception, it devolves upon a complete philosophy of mathematics to show precisely what that relation is, and the exact dialectic by which the conception arises in the process of intelligence, and unfolds itself into such phases as necessitate that general character of combined law-discovery and calculation which we have seen belongs to mathematics. In fact, a failure either to ascend from the correct general description of what mathematics *does*, through all intervening conceptions, to its more generic expression in terms of the supreme conception Quantity, or else to descend from this conception through the proper logical intervals to the natural development of the definition-by-what-is-done, would seem to shut out the subject from the realm of strict science altogether, and to represent it as being a science only in virtue of being a systematic *art* — an art, that is, whose rules can be stated in methodical order. Now, it appears to be an instinctive conviction that mathematics is, on the contrary, a science in

the severest sense, namely, of a system of intelligence pervaded by a sovereign Idea — a conviction which I shall presently attempt to verify by analysis — and this can never be satisfied by any treatment of the subject which reduces its fundamental character to that of an art, no matter how methodical the art may be. Moreover, even should we have to admit that a *complete* dialectic of mathematics, as a phase in the entire science of Quantity, has thus far not appeared by reason of our inability to define Quantity itself, or to exhibit its Rise out of the Prior Elements of Intelligence, still a philosophy accurate as far as it reaches should be competent to show *why* "measurement" is the problem of mathematics, and *why* it is accomplished by indirection—an indirection, moreover, not arbitrary, but coherent according to a determinate system; it should be competent to show just what "measurement" logically is, by exhibiting how it arises out of the interaction of the logical constituents of Quantity itself, as these are taken-up-again into intelligence. That Comte's definition does not even attempt this, is due to the very nature of the Positive Philosophy : he would unquestionably claim that he had finished his task when he was able to assign a description of mathematics so generic as to include all the facts of its use that come within the range of sensuous experience; but we have only to contrast this result with what a searching intelligence asks, as pointed out above, to appreciate the inherent deficiency of a method whose sole convincing evidence is a generalized experience of the natural world.

And this deficiency displays itself more clearly when we come to estimate the detail of Comte's classification. First of all, granting that mathematics, as a method of measurement by means of functions, does naturally fall into the two departments of *creating* and of *computing* functions, with what propriety is the one called *Concrete*, and the other *Abstract?* Surely not because the laws established in the first are in any sense more specific, or more definitely comprehensible, than the rules of computation applied in the second; surely not because a law of algebraic transformation is any more vague or filmy than a law of Mechanics or a proposition in Geometry ; surely not because the rules of computation are of more sweeping generality than the laws of motion or

the properties of figure. The rule that one and one make two is indeed necessary in its truth, and universal in its application, wheresoever we find one unit to combine with another, whether in the world of nature or in the universe of thought; but the law that the total motion is jointly proportional to the velocity and time, is likewise necessary in *its* truth, and universal in *its* application, no matter where we find a uniform motion, be it in the finite world of material bodies or in the infinite world of conceivable atoms; and so, also, the theorem that the square upon the hypotenuse of a right triangle is equal to the sum of the squares upon its other two sides, is necessary in *its* truth, and universal in *its* application, no matter whether we consider any right triangle that may occur on the earth, or any of the infinitude of right triangles that we may choose to imagine. Thus the laws of the creative and computative departments of our general Theory of Functions are alike, and equally, universal and necessary: otherwise, indeed, they would not be *laws:* and they only differ in their subject-elements, the former class being laws of *motion* or of *form*, and the latter laws of *number*. Not on the ground, then, of the ordinary and natural distinction in the meaning of those terms, can we call the one class concrete, and the other abstract: let us, therefore, try the distinction which Comte himself proposes. "The first class," says he,* "should be called *concrete*, since it evidently depends on the character of the phenomena considered, and must necessarily vary when we examine new phenomena; while the second is completely independent of the nature of the objects examined, and is concerned only with the *numerical* relations which they present, for which reason it should be called *abstract*." Now, when we reflect that the entire computative side of mathematics is simply a system of transforming and valuing functions by applying to them the fundamental properties of number, Comte's last statement is seen to ignore the manifest fact that his second class is concerned with laws quite as much as his first—with laws, too, that quite as much depend upon the character of the "phenomenon" *number* as do those of Mechanics and Geometry upon the nature of *mo-*

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* *Cours de Philosophie Positive*, tome 1, p. 132.

tion and *form*. Moreover, the generic laws of motion and form do *not* vary with the phenomenon considered—unless we mean by this statement, that the laws change when we pass from the province of motion to the province of form, or that the laws of uniform motion differ from those of accelerated motion, and those of rectilinear figures from those of curvilinear; and even when we recognize this last obvious fact, and contrast the variety which the laws of motion and form present, with the almost absolute uniformity of the laws of number,* the truth is not expressed by saying that the laws of number are *abstract* as compared with those of motion and form, but by saying that they are more *simple* and *uniform*. In short, Comte's statement sums itself up in the clear proposition, that the laws of motion are one thing, the laws of form another, and the laws of number .still another: which can hardly be gainsaid. But the only warrant that this can give for calling the laws of motion and form *concrete*, and those of number *abstract*, lies in a somewhat plausible play upon words: we may fancy that we cannot conceive of the former except as *mixed up* (*concretas*) with the perpetual flux of physical phenomena, whereas the latter are readily and usually conceived of as entirely *withdrawn* (*abstractas*)†
from such a commixture. The plausibility, however, is superficial and deceptive: for, as above shown, the laws of motion and form are as completely separable in thought from the phenomena in which they appear to our senses as are those of number, both being universal and necessary; and, on the other hand, were it true that we only reach the laws of motion and form by generalizing upon the facts observed by the senses, consistency would require us to maintain that all the laws of number, even the fundamental axioms, are reached in the same way;‡ for nothing is more obvious than

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* The continuity of the laws of number is only broken by the distinction between *commensurable* and *incommensurable* numbers.

† Thus, we have in our common Arithmetics the time-honored title of *abstract numbers*. The phrase is inaccurate, but nevertheless serves well enough to call up the important truth, that numbers may be taken both abstractly and in the concrete.

‡ This is what Mill actually does maintain. See his *System of Logic*, v. i., p. 260.

that numbers and their combinations are everywhere existent in the natural world, and are therefore a perpetual element of our daily experience.

It thus appears that the attempt to distinguish as *concrete* the establishment of laws of functional derivation, and as *abstract* the application of the properties of number to the actual derivation of functions, is essentially unscientific. The distinction breaks up under the light of reflection: what is called concrete may just as well be called abstract; what Comte calls abstract may, upon his method, just as well be called concrete; or, rather, *should* be called so, thus convicting him of inconsistency in the use of his own principles. And such must of necessity be the result of every attempt to set up as categories of science the Abstract and the Concrete —those twin Phantoms which reflection conjures up to aid it in its progress toward the comprehension of real existence, but which make no part of the abiding truth of that existence, into whose living completeness they are both absorbed.

The unscientific character of Comte's distinction of mathematics into Abstract and Concrete, shows itself in a second defect of his scheme, namely, in the subdivision of the Law-forming department into Geometry, Mechanics, and Thermology, with the implication that the subdivision is in a sort of suspense, awaiting an indefinite expansion by the discovery of new quantifying "laws" in the departments of Physics. The notion of a "suspended" science is of course self-contradictory; yet into this contradiction Comte directly falls, in conceiving of the laws of form and motion as having a "concrete" character merely : that is, as founded solely upon Induction, and therefore of course inviting into their own rank the results of induction known as the "laws of heat." For, in consistency, not only the laws of heat, but those of light, and sound, and color, and magnetism, and electricity, and chemical affinity, and vital force—the laws, in short, of the whole circuit of the Correlation of Forces—should be advanced into the same coordination. In fact, there should be no limit to the number of sciences coördinate with Geometry and Mechanics, except the limit fixed by the possible number of physical phenomena and the power of the human mind to discover their "laws." And since physical phenomena exist of necessity in an infi

nite progression, the possible subdivisions of "Concrete" mathematics should be infinite in number.

This defect in Comte's classification appears to arise from his failing to distinguish clearly between the *essential nature* of a mathematical law, and the *application* of it to the material world; a failure very liable to occur in the attempt to put Mechanics into its proper place in the entire scheme of mathematics, as the treatises upon that branch do not as yet dissever the laws of rest and motion from the equilibrium and motion of *bodies;* so that, as usually discussed, it is more an applied mathematics than a speculative. Nevertheless, the distinction exists; and it is only under the aspect of their *essential nature* that the laws either of form or of motion can have any place in the *science* of mathematics; moreover, the so-called "laws" of heat and light, and so on, are only specific cases of universal laws of uniform or accelerated motion. It is the more remarkable that Comte should insist upon confounding these two aspects of mathematical law, as he has elsewhere clearly discriminated between a science and its application,* and as it must be on the ground of such a distinction that he refuses a place in his scheme—as he does— to Optics, to Acoustics, and above all to Astronomy, which, in its phase of Celestial Mechanics, is assuredly the crowning glory of applied mathematics. On the other hand, had he held consistently to the distinction between science and its application, Applied Mechanics and Thermology would have fallen into their proper place in line with Astronomy, Optics, and Acoustics; and the prospectively endless succession of coördinate branches would have been a legitimate result; for, as Bacon† long ago remarked, "The greater increase Physics receives, and the more new principles it develops, the more will Mathematics be called into new applications, and the more numerous will the *Mixed Mathematics* become."

There is one other point in Comte's treatment of his " Concrete" subdivision which seems to me defective : I mean his discrimination between Pure and Analytic Geometry as " Special" and " General." After very justly criticising the

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* *Cours de Philosophie Positive,* tome 1, p. 345.

† *De Aug. Scient.,* lib. iii., cap. 6.

common practice of calling the former *Synthetic* and the latter *Analytic*, he puts his own distinction upon the following ground :—The ancient Geometry can discuss no problem of form or dimension except by taking up all the different figures, rectilinear and curvilinear, one by one, and obtaining a separate solution for each order of figure, triangle, circle, conic, spiral, or whatever else it may be, the *method* of solution having to be invented *new* for every new figure taken up; whereas, in the Geometry of Descartes, problems of contact, of curvature, of asymptotic approach, of singularity in flexure, of the length of the subtangent and subnormal, of rectification, of quadrature, of cubature, are solved in formulæ of absolute generality, which apply to all figures alike. Now this statement, which the mathematical reader will at once recognize as entirely correct, seems at first sight to justify Comte's nomenclature completely; but a closer examination will convince us that it cannot serve to separate Geometry into its scientific provinces, inasmuch as it after all misses the point of *characteristic* difference between Analytic Geometry and Pure. The *method* of Analytic Geometry is undoubtedly more general than that of Pure Geometry; but so is the method of Algebra more general than that of Arithmetic; and yet, as Comte himself has noticed,* it would now be a grave error to define Algebra, with Newton, as a *Universal Arithmetic*, thus setting up Arithmetic as the Special Calculus, and Algebra as the General Calculus. The truth is, that having set out to find the departments of mathematics according to their *object-matter*, it is a formal blunder to introduce a new principle of division, and distinguish between the *methods* according to which a given object-matter is discussed. On the contrary, to raise the so-called Pure and Analytic geometries into sub-departments of Geometry-in-general, we must show that they deal with distinct provinces of its whole object-matter; and, more specifically, admitting that its whole object-matter is *the functional laws which connect the several parts of figures with each other*, we must show that Pure Geometry deals with one class, or else rank, of those functional laws, and Analytic Geometry with another. For this exposi-

* *Cours de Philosophie Positive*, tome 1. p. 177.

tion, we are not ready at this stage of our discussion; but, at a later one, it will appear that Pure Geometry deals with those functional laws which determine *magnitude*, and Analytic Geometry with those which determine *form*.

Passing over, now, to Comte's treatment of the Computative side of mathematics, the chief matter for criticism is his mode of distinguishing the Higher Calculus from the Lower. By defining the former as the *Calculus of Indirect Functions*, and the latter as the *Calculus of Direct Functions*, he grounds his distinction not on the nature of functions as such, but on that of the phenomena in whose "laws" he supposes the functions to originate : for by a *direct* function he means* a quantifying law connecting the parts of phenomena themselves; and by an *indirect* one, a law connecting not the parts of the primary phenomena, but certain auxiliary quantities, united with the primaries by uniform laws of derivation, and substituted for them because *their* laws are too complex to be put into equations directly. Now this mode of distinction, which at first sight seems to gain support from Lagrange's great generic insight of regarding the auxiliary infinitesimals of the Higher Calculus as algebraic DERIVATIVES of the primary functions, has in it, beyond question, this superficial plausibility : it does appear to lift the mind contemplating the "differentials," which play so essential a part in the Higher Calculus, into a field of view which surprises by its unexpected and apparently endless extent; for we seem to see these obscure analytical elements suddenly shining in the wide horizon of the conception of *"auxiliary quantities derivable from their primitives according to any law whatever,"* and the present Calculus of Differentials, vast as we thought its generality when compared with ordinary Algebra, appears in the yet grander rôle of one specific method out of an infinite system of possible methods for deriving auxiliary functions. It must be observed, however, that all this splendor of generality is due to Lagrange's conception that differentials are DERIVATIVES, and not to Comte's transcription of that great theme into the notion of Indirection; so that it has nothing to do with viewing differentials as capable of simpler

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* *Cours de Philosophie Positive*, tome 1, pp. 189, 194.

relations than the primitive phenomena, except in so far as this notion of "Indirect Functions" borrows the idea of derivation itself. In fact, the formation of functional relations between differentials on the ground of their superior simplicity, belongs not to the *theory* of the Higher Calculus, but to certain of its *applications;* for instance, to rectifications, quadratures, cubatures, and the determination of maxima and minima; in which we certainly do establish some very obvious differential equations for the express purpose of passing from them to integral equations we could not readily obtain by considering the integrals themselves. It is not the procedure of science, however, to define a theory by its applications, particularly by a portion of them. But, above all, this resting the distinction between the two departments of analysis upon the difference-in-complexity of the phenomena to which they are applied, to say nothing of its resemblance to the boy's subdivision of Arithmetic into "*Arithmetic with Easy Sums*" and "*Arithmetic with Hard Sums*," is unscientific because it carries over considerations from Comte's "concrete" field to subdivide his "abstract" one; whereas, if the latter is a real province of mathematics (and, in so far as it is characterized by the idea of *computation*, it certainly *is* a real province), it ought to subdivide *itself*, and *by principles involved in its own idea*. In short, if Analysis *is* the Calculus of Functions, and if there *is* a difference in object-matter between the Lower Calculus and the Higher, that difference must be sought in the nature of the *different orders of computation* which are employed in the two.

And, at this juncture, the root of Comte's inaccuracy comes plainly to the surface: his error grows naturally out of a vagueness in his conception of Analysis itself, and a consequent confusion in his view of the relations between Algebra and Arithmetic. At first, he correctly ranks Arithmetic and Algebra (using this term, in its widest sense, as equivalent to *Analysis*) as the two coördinate branches of the general Calculus (or Science of Computation), defining the former as the *Calculus of Values*, and the latter as the *Calculus of Functions*. But this inexact characterization of General Algebra presently leads him to bring back Arithmetic (which he sees may also be regarded as a calculus of functions) into the prov-

ince of this same Analysis; and he ends at last with the singular contradiction of coördinating Arithmetic with Algebra taken in its ordinary and very restricted sense of a *method for solving equations,* presenting the two as subdivisions of his "Ordinary Analysis" as distinguished from his "Transcendental." This confusion is heightened by the fact that he had, at the outset, in a single passage,* undertaken to explain his phrase *Calculus of Functions* by defining General Algebra as the science having for its object "the transformation of *implicit* functions into equivalent *explicit* ones": a definition which, so far from reaching the comprehension of Algebra as inclusive of all Analysis, is in fact the exact description of *ordinary* Algebra; that is, of the Lower Calculus as distinguished from the Higher. But had he held to the road which leads logically from his starting-point, he would have continued to coördinate Arithmetic with Algebra in its most extended sense; he would thus have recognized that the computation of functions involves their *transformation* and their *evaluation,* and might have defined General Algebra (or Analysis) as the *Calculus of the* FORMS *of Functions,* and Arithmetic as the *Calculus of the* VALUES *of Functions.* And having thus arrived at *transformation* as the essence of all analysis, his path would have been clear to a real discrimination between the Lower and the Higher Calculus, on the ground that they involve two distinct *orders of transformation.*

I need not more than advert to the additional fact, that this confusion as to the nature of General Algebra accounts for Comte's imperfect elucidation of the relation borne by the Calculus of Variations to the rest of the science. Lacking the conception of Analysis as a general method of transformations, he contrasts the Calculus of Variations with the ordinary Integral Calculus; or, rather, treats it as a modified form of that Calculus; whereas, in the light of the idea that functions are *laws* or *forms* of derivation, it ought to be contrasted with the whole calculus of which the Differential and the Integral are the branches, and defined as the *Higher Calculus of* VARIABLE *Functions,* while they in their correla-

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* *Cours de Philosophie Positive,* tome 1, p. 187.

tion are defined as the *Higher Calculus of* CONSTANT *Functions.*

With these considerations, we may now leave the labors of Comte, and pass on to essay the actual construction of the System of Mathematics, as it evolves itself in the general procedure of Intelligence.

III.—CONSTRUCTIVE.

We start, then, with the truism that mathematics is a *form of intelligence;* and it is our first business to find the definition of it, by discovering what it is that intelligence is there endeavoring to do.*

And, without venturing to attempt the definition of QUANTITY, let us avail ourselves of our familiarity with that concept as an inseparable factor in all our intelligence; let us also take advantage of our experience concerning the use of mathematics; and we shall be warranted in the provisional statement that *mathematics is that form of intelligence in which we bring the objects of the phenomenal world under the control of the conception of quantity.*

This is vague enough, to be sure; and, were we to stop at this, we should have no real definition. But it is the beginning, from a point commanding universal assent; and, even in this its empty abstractness, it has this immediate advantage:—we see at once that the System of Mathematics must subdivide itself as *Science* and as *Art;* since, as form of intelligence, it must be science, and, as applied to the phenomenal world, it must be art. We thus take a long step toward clearing up the confusion of its multitudinous branches, by resolving the whole into two grand divisions of SPECULATIVE and APPLIED mathematics.

Thus, under the head of Applied Mathematics (the relation among whose subdivisions is determined immediately by the nature of the phenomena with which they deal, and therefore calls for no dialectic), we at one dispose of the greater part

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* In thus setting out to fix the system of mathematics by its *motive*, we only obey the controlling principle of all the sciences; since they all, as forms, or *phases*, of intelligence, must have their *raison d'être* in the aims of intelligence— except, indeed, the Objective Logic, which, as the Science of Intelligence itself, is its own *raison d'être.*

of the long list with which we perplexed ourselves near the opening of this article. We may remove, then, as requiring no further discussion: Astronomy, with its subdivisions, Celestial Mechanics and the Calculus of Observations; Surveying—Geodesic, Topographic and Mensurative; Navigation—Celesto-spherical and Plane; Acoustics; Optics; Thermodynamics; Molecular Mechanics; Physico-Mechanics, branching into Statics and Dynamics, Hydrostatics and Hydrodynamics; and Practical Mechanics, including Architecture (with its subsidiary art of Stereotomy) and Engineering in its various branches, Civil, Military, Naval, and Mechanical; not forgetting that these last involve the mathematical principles of Mining, Bridging, Road-building, Fortification, Gunnery, Ship-modelling and Armoring, and of all the parts of Mill-work, whether motors or machines. Any further grouping or defining which these branches may call for, in order to render clear their mutual relations, will sufficiently appear in the tabulated scheme at the close of this article; so that the sketch of this grand division of mathematics may be completed by merely recalling the fact (noticed on a preceding page), that the number of its subdivisions is potentially unlimited, and will continue to increase with man's increasing knowledge of the natural world.

Turning now to Speculative Mathematics, we have in this side of our subject the essence of our whole problem of classification; for the comprehension of a form of intelligence as *science,* is the necessary and sufficient condition of its definition and subdivision. If, then, we can reach the comprehension of Speculative Mathematics—that is, if we can trace in *that* the exact procedure which intelligence makes under its ruling concept Quantity—we shall be able to replace the emptiness of our provisional description by a full and living definition. Let us, then, attempt the exposition of the mind's continuous descent from the concept of Quantity to a science of Mathematics.

A. *Process of finding the Thought-Constituents of Mathematics.*

SPACE and TIME—these are the logical *sine quâ non*—the necessary thought-element—of all phenomenal existences; just as the air is the indispensable matter-element of our

vital existence. Into these as such thought-element, there-
fore, must intelligence project its concept Quantity, when it
seeks to dissolve the phenomenal world by means of that.
But Space and Time react upon the projected concept, so that
it is itself dissolved; only to rise, however, into a threefold
power for intelligence, as Extension, Motion, and Number;
for these three "modes" of quantity are simply the thought-
phases into which it breaks up under the reaction of Space
and Time. Quantity projected into Space considered aloof
from Time, is *magnitude;* projected into Time considered
aloof from Space, it is *duration;* and these two abstractions
disappear in the wider one of EXTENSION, which may there-
fore be defined as *Quantity projected upon the abstract sever-
ance of Space from Time.* But if we let go of our abstraction,
and consider Space and Time in their *real* aspect of *coëxist-
ence,* the concepts magnitude and duration flow-together and
annul-each-other in the idea of MOTION, which we may accord-
ingly define as *Quantity projected upon the concrete interflow
of Space and Time.* Extension and Motion, however, only
present Space and Time in their phase of *infinite continuity,—*
a thought with which the Conceptive Understanding is en-
tirely unable to cope. To that side of intelligence, nothing
but the *finite —* the *externally* limited — is comprehensible;
and yet, since it *is* a side of intelligence, its protest against
Quantity as pure Continuity must have a hearing. Thus,
when this Conceptive Understanding " sinks exhausted" be-
fore the thought of unbeginning and unending extension, or
of unarising and unceasing motion, and is on the point of
rejecting both as meaningless abstractions, the Thinking Rea-
son comes to its rescue: it negates the continuity of Space,
and posits the *point;* it negates the continuity of Time, and
posits the *instant;* in the infinite of Space, it finds a *Here* and
a *There;* in the infinite of Time, a *Then,* a *Now,* and a *Here-
after.* Between these, as limits, the whole mystery of Exten-
sion is resolved for the Understanding into a *limited extent*
on the one hand, or a *period elapsed* on the other; the whole
mystery of Motion, into a *distance traversed in a given time.*
Thus Quantity descends into finite terms; and, what is more
to the purpose, in doing so passes to its final and most intel-
ligible phase. For when in Infinite Continuity we set up

limits, then, in so far as it is Extension, its unity breaks up into *multitude;* and, in so far as it is Motion, it breaks up into *succession;* but the union of these two constitutes the idea of NUMBER, which may consequently be defined as *Quantity projected upon the annulled-continuity of Space and Time.*

Such is the genesis of Extension, Motion, and Number,— the three elements of thought in whose correlation, as we shall presently see, mathematics finds at once its occasion and the ground of its method.

B. *Construction of the Fundamental Problem of Mathematics.*

NUMBER, from the manner of its logical origin, as that has just been unfolded, is seen to be the triumph of the Conceptive Understanding over the primordial chaos of Space and Time; or, more truly, the reconciliation of the Understanding, which can comprehend nothing but finites, to the infinite continuity of Quantity as seized by the Reason in pure Extension or Motion.   It is the instrument for completely simplifying these less determinate phases of Quantity, from which the Understanding naturally revolts as from a bewildering mystery; for, as Number arises out of dissolving Extension and Motion into successive *parts*, it becomes at once their interpreter to the understanding, by presenting all *periods elapsed*, all *limited extents*, and all *distances traversed in given times*, as aggregates of equal parts as small as we please and therefore as easily conceivable.   Hence there arises in the mind a persistent tendency to convert all forms of Extension and Motion into Number; and the three are so correlated that this tendency is readily satisfied, and becomes the occasion of that systematic contrivance for effecting this conversion which we call mathematics.   Thus it appears that the Fundamental Problem of that science, stated in its universal form, is this: *To pass at will from the mental province of Extension or of Motion to that of Number.*

The general, abstract solution of this problem is exceedingly simple, being in fact given in the very process by which Number arises: we have only to resort to the easy expedient of comparing Extension or Motion with any one of their equal parts, and then considering the "ratio" of the whole to this

arbitrary unit. And here, in our universal language, "measurement" translates itself into "converting extension or motion into number": we see what "measurement" logically *is,* and why it is the problem *par excellence* of mathematics. It remains to discover why it must be solved by "indirection"; that is, by deriving the numerical equivalents of extensions or motions through a complex *system* of relations, instead of by directly applying the unit-part to the measurable whole.

c. Origin of System in the Procedure of Mathematics.

The path which leads us to the discovery last mentioned is not long; we perceive, indeed, with but brief reflection, that Space and Time baffle the attempt to apply the expedient of the direct unit as easily as they repel that of bringing their pure continuity within the comprehension of the imagination. In fact, it is the persistence of that infinite continuity, despite its apparent cancellation by arbitrary limits, which effectually annuls the petty device of our measuring unit: in the infinite possibilities of Space and Time, extents, durations, and motions run on forever, and pass beyond our reach when we attempt to lay our measure upon them. Could we actually annul Time, we might actually measure Space by following its successive extensions; could we actually annul Space, we might follow out the successive epochs of Time; but, fortunately, we can really do neither: the two elements *coëxist* in the real world—the actual application of the unit to the whole is, as a universal or even general procedure, an impossibility, and we are forced to abandon this conceptual method and take refuge in the mightier powers of the systematizing Reason. For though Space and Time baffle us when we attack them from the side of our finite existence, they yield at once when we bring against them the idea of Self-Relation. Self-related points we can posit in space; self-related instants, in Time; and out of these arise, on the one hand, Form and Figured Extension, and Rate (or Velocity) of Motion, on the other. To determine the form, or figure, of an extension, is necessarily to determine the magnitude of all parts that can be set up within or about it in accordance with given conditions of position; to determine the rate of a movement, is necessarily to determine the distances traversed by

it in known times, or the times due to known distances. And if, armed with these new resources, we turn to the idea of Number, and observe *its* inseparable properties—how it exists in a ceaseless flow and re-flow; how we can count forward by unity from zero to any number conceivable, or backward by unity from any number to zero again (*addition* and *subtraction*); how, by seizing as unity any number of units, we can sweep by this greater stride to any desired other number (*multiplication*); how, by reversing this process, we can lessen as much as we please the number of backward steps from any number to the original zero (*division*); how we can count any number, taken as unit, as many times as there are units in itself, repeating this as often as we please, (*involution*); how, finally, we can undo *this* process as well as the simpler ones (*evolution*)—we see that all these so-called "operations" of addition, subtraction, multiplication, division, involution, and evolution, also result from the idea of Self-Relation*; the mutual adaptation of the processes of Number, and the interdependences of Figured Extension or Rated Motion, present themselves clearly to our anticipation; so that we return again to the provinces of Form and Movement, in the conviction that the magnitudes of the parts of figures, and the quanta of the constituents of motions, are connected with each other by discoverable *numerical* laws; in other words, that these magnitudes, and these quanta of motion, *grow out of each other at rates expressible by the very "operations" constituting the essential properties of numbers.* And this anticipative conviction the contents of Mechanics and Geometry completely confirm; for all the theorems of Motion and of Figure are, either directly or indirectly, just such statements of the numerical combinations that have to be made of constituent motion-elements or magnitudes, in order to produce certain derivative motion-elements or magnitudes. Thus it is that the mind, seeking to solve the problem of converting Extension and Motion into Number, finds itself possessed of a vast and coherent *system* for effecting the solu-

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* Numeration is possible only by this :—that the One, going-out-of-itself into the Many, shall again return-into-itself as the One (of Quality) which brings the Many into being Successive Units : we cannot count the *mere* Many. Thus, the properties of Number arise out of the system of the Self-related Unity.

tion,—a system of *laws connecting change with change,* *— a system for *deriving* values from given ones of any nature by the very modes of combination which constitute the laws of Number itself, and by which numbers are themselves derived from each other.

D. *Definition of Mathematics considered as Science.*

We have now reached the point from which the definition of mathematics, in its aspect of science, comes clearly into view. It may be stated as follows:

Mathematics is the science of converting Figured Extension and Rated Motion into Number, in accordance with the system of numerical laws which connect the parts of figures and the elements of motions with one another.

Or, inasmuch as these numerical laws of mutual dependence constitute the mentioned parts and elements *functions* of each other, we may state the definition more concisely as follows:

Mathematics is the science of the functional laws and transformations which enable us to convert Figured Extension and Rated Motion into Number.

E. *Subdivision of the Science, and Coördination of its Parts.*

The science which actually succeeds in passing over from Extension and Motion to specific Number, and that, too, by means of a System of Functions, will of course involve two grand movements: the investigation of the numerical laws which connect the parts of figures and the constituents of motions with each other; and the working out, from the relations given by these laws, of the actual numerical values of the magnitudes or motion-elements sought. Thus the entire field of mathematics divides itself into two main provinces: the one, a Body of Doctrine concerning the numerical combinations which have to be made of given parts of figures, or given parts in a moving system, in order to obtain required parts; the other, a Theory of Operations by which the doctrines of the first may be represented, transformed, and finally

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* Sir W. R. Hamilton, in preface to his *Algebra as the Science of Pure Time.* See the *Transactions of the Royal Irish Academy,* vol. xvii.

evaluated by means of the processes which we call the essential properties of Number. It is our business, in the first, to *create* the System of Functions; in the second, to *transform* and *evaluate* its members when created. In short, mathematics consists of (I.) a Mathesis, and (II.) a Calculus.

I. The Mathesis of mathematics, again, has two subordinate provinces, *Geometry* and *Mechanics*, depending on whether the functions created are laws connecting the parts of *Figures*, or the constituents of *Movements*. And these two* are *all*, whether actual or possible; inasmuch as, excluding Number, Extension and Motion exhaust the possible modes in which Space and Time can react upon Quantity, as has already appeared at an earlier stage of the present discussion.

1. If we define *Geometry* as the *Science of Figure*, or, more exactly, as the *science whose object is to determine what numerical combinations connect the parts of figures with each other*, we shall not advance far in the detail of verifying the definition by the actual content of the science, without noticing that the theorems of Pure Geometry present two quite distinct classes: by far the greater part of them certainly do connect *magnitudes* by numerical relations which enable us to find one when others are given; but another part, at least equally striking, if much less numerous, seem to be altogether concerned with relations of *position*, that is, with *form:* such are the theorems that *Any three points are always in the same plane;* that *Any three points, not in a straight line, are on the circumference of a fixed circle;* that *A point whose distances from a fixed point and a fixed straight line are in a constant ratio is on the periphery of a conic;* etc. In pure geometry, these do not seem to express relations of magni-

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* Should the curious reader here ask—What has become of *Duration*, that only *Magnitude*, as befunctioned by Figure, is taken to cover the whole science of Extension?—I reply that the quantification of Time is so simple as to make no science, its whole mathesis consisting in the law which connects a *period elapsed* with its limiting *dates*. Thus, its sole function is $P = D - D'$: whence, $D = D' + P$; or, $D' = D - P$.

I submit this answer with reserve, however; remembering that the illustrious Hamilton has attempted the construction of a *Science of Pure Time*, and its absolute identification with the whole of *Algebra*, taken in the widest sense. See the *Transac. Royal Irish Acad.*, vol. xvii.

tude, nor to yield any functions by which magnitudes can be calculated. But, on the one hand, they always appear in that branch of geometry as *subsidiary steps*, leading to the proof of other theorems which *are* functional, and this seems to be their only scientific relation in that branch; while, on the other hand, the distinction which they suggest between *theorems of dimension* and *theorems of form* has the most essential significance in settling the real difference between the Ancient Geometry and that of Descartes. For, in truth, it may be said that the so-called Pure Geometry is unable to reduce the conception of Form to the functional relation; that its principles and methods are only adequate to the task of befunctioning magnitudes by boldly taking Form for granted; and that the solution of the more general problem, of expressing Form itself in the idea of a function, is only reached in Analytic Geometry, where, by the simple but universally-sweeping "Convention of Coördinates," the Form of every conceivable curve or surface is brought within the conception of a relation between magnitudes, and accordingly represented by an equation. And thus we learn that Geometry in General breaks up into two sub-sciences — *Pure Geometry*, whose functions are those of *Magnitude* merely; and *Analytic Geometry*, whose functions are functions of Form.

Within the province of Pure Geometry we have the subordinate one of *Trigonometry*, in which the theory of the functional character of geometric relations is simply carried out to the last detail in respect to the calculations of the parts of a triangle. But closely allied to this Functional Pure-Geometry, there exists another, antagonistic in idea, whose aim is to carry geometry out of relation to the computative side of mathematics, and set it up as a self-contained science. I refer to what may be called *Constructive Geometry*, with its subdivisions, *Graphics* and *Descriptive Geometry*. The object of these is, to determine by a proportional diagram, whose unit is taken at will, the total linear or superficial value of a required part of a plane figure or a solid, without calling in the aid of any calculation whatever—the required part being taken directly from the drawing (which has been carefully constructed to express the determining conditions), and compared with a standard physical unit. This method has certain ad-

vantages, being sometimes rapid as compared with calculation, and is to a certain extent exceedingly useful, as in carpentry and stone-cutting; but it is inaccurate, from the necessary imperfection of drawing-instruments, and is limited to a comparatively small number of theorems. The method of *Graphics* — which is the constructive solution of *plane* problems—may be described as *direct:* by it, the required part *itself* is drawn and measured; *e. g.*, the line which forms a geometric mean between two given 'ones; or, the square equal to the sum of any number of given squares. The method of *Descriptive Geometry*, on the contrary, is *indirect:* its object is the construction of the parts of *solid* figures, which cannot be directly drawn on a flat surface: hence the brilliant invention of MONGE, by which we substitute, for the parts of figures themselves, their *projections* (or *shadows cast from an infinite distance*) upon auxiliary planes; these projections once constructed according to a systematic method which Monge has developed, we readily pass from their dimensions to those of the actual figures, with which they are connected by a uniform and very simple functional law.*

The system of Graphics presents itself in three stages:— *Lineal Geometry*, the comparatively recent invention of LAMBERT, in which the constructions are effected with the help of the ruler alone; *Geometric Construction* technically so-called, in which we employ only the ruler and compasses; and *Mechanical Construction*, in which curves of higher orders than the circle are employed, and are generated by means of special instruments which embody some one of their defining properties.

2. The term *Mechanics*, as used to designate the second province of the Mathesis, must be understood as strictly confined to the necessary *a priori* laws of motion and moving systems, and as therefore excluding the vast body of physical considerations and contingent conditions which constitute so large a part of Mechanics as ordinarily understood; indeed,

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* The careful reader can hardly fail to notice that the above description of the bearing of Graphics and Descriptive Geometry upon the whole science, is essentially the same as Comte's. Comte, indeed, appears to me to have seized the exact philosophy of this particular episode in his subject, and his treatment of it seems the happiest he has reached.

it might be well to replace it by the term *Kinematics,* were this not so novel and awkward in sound that the mathematical public does not seem to take to it kindly; it is an objection, too, that the new title seems to shut out the consideration of forces in equilibrium. And, in fact, we shall better define Mechanics as the *Science of Force* than as the science of Motion; because, in order to cover systems of motion with entire generality, and so include the system in equilibrium, we must seize Motion as *potential* rather than *actual,* that is, as Force.

The well-known subdivisions of Mechanics—Statics and Dynamics—are thus provided for. In the former, we investigate the laws of Equilibrium, which pass into the functional form by the process of resolving forces into their components; in the latter, the laws of actualized Motion, whether simple or systematic.

II. Crossing now to the CALCULUS, or the computative side of mathematics, it is plain that here, too, there is a great two-fold division: assuming that the Mathesis furnishes to our hand the numerical laws which connect required values with given ones, these laws must be expressed in proper symbols —in a word, they must have a convenient *algorithm*—and, when so expressed, the resulting equations (for a numerical law cannot be expressed except in an equation) may require transformation of different orders before they become available for the actual calculation of the values required; finally, when they *are* so transformed, the fundamental processes of number must be applied to them, and the required value thus actually found. Hence the Calculus breaks up into (1) the *Transformation of Functions,* and (2) the *Evaluation of Functions.*

For, defining a function as a *number derived from others by certain numerical combinations, so that its value varies with theirs,* we must distinguish between its *value* and what may aptly be called its *form.* By its *value,* is meant its *place in the general scale of numbers,* or the *result* of the operations by which it is derived from its primitives ; by its *form,* the series of *operations themselves* by which it *is* derived. Thus, in the familiar equation $y = \sqrt{2rx - x^2}$, which connects

the length of the perpendicular dropped from the circumference upon the diameter with that of the two segments into into which it divides the latter, the *value* of the function y is the number for which y stands; while its *form* consists in the operations by which that number is derived from the number x, namely, multiplying x by the diameter $2r$, subtracting the square of x from the product, and extracting the square-root of the remainder.

In view of this distinction, then, we may define *Arithmetic* as the *Science of the Evaluation of Functions;* and to the *Science of the Transformation of Functions* we may, in accordance with the uniform usage of such philosophic mathematicians as LAGRANGE and HAMILTON, give the name of *Algebra.**

1. Of *Arithmetic* it needs only to remark, that this branch includes the general Theory of Numbers, and the methods of rapidly combining, in the so-called "fundamental operations," the essential properties developed by that.

2. To render clear the grounds for subdividing *Algebra* into a Lower and a Higher Calculus, we must distinguish between the two states, *implicit* and *explicit*, in which the form of a function may exist, and the two classes, *commensurable* and *incommensurable*, into which the profoundest analysis finally resolves numbers. For, on the one hand, a function as brought to us from the Mathesis may be mixed up in various terms of the equation which connects it with its primitives, and then its form is only *implicit;* or it may stand alone and clear in the first member of the equation, while its primitives in their due combination make up the second, and then its form is *explicit*. And, on the other, just as numbers are primarily conceived as parted from each other by a constant finite interval called *unity*, and are in general simply exact integral or fractional multiples of that, while yet we everywhere find numbers, like 2 and 3, whose *ratio* (or *relative* complex) of this unity we can never find, but can only approximate endlessly; so, in attempting to reduce Extension or Motion to Number, by the aid of an arbitrary unit of their own species, we may

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* Lagrange, *Leçons sur le Calcul des Fonctions*, leçon 1; Hamilton, *Algebra as the Science of Pure Time*, Transac. Royal Irish Acad., vol. xvii.

and often do come upon quantities that result only in these *incommensurable* numbers; *e. g.*, the ratio of the circumference of a circle to the diameter. The secret of such numbers seems to be, that, from the very nature of Number in general, there is no limit to the smallness of the " unit," and these incommensurables simply push the mind back again upon that *continuity* for whose cancellation the Understanding so eagerly seized upon the contrivance of the " unit," but which persists through all as the essential principle of quantity itself. Since, then, the comparison of the parts of Extension and Motion must inevitably lead to functions expressible only in terms of incommensurables, it becomes an essential to the perfection of the calculus that we shall be able to transform such relations between incommensurables into corresponding ones between commensurable, or ordinary discrete numbers. Moreover, as in comparing two incommensurable magnitudes, such as the diameter and the circumference, we endlessly approach the true ratio as we go farther and farther on in the process of measuring the greater by the less, the less by the remainder, the remainder by the new remainder, and so on; we see that the logical unit of this tantalizing ratio is an evasive, endlessly diminishing unit—an *infinitely small element* of the magnitudes compared—in a word, a *differential.* Hence, the added necessity that our calculus should enable us to pass readily from finite quantities to these their infinitesimal elements.

In receiving, then, a function to be transformed, it may either be our object to devise a method of passing at will from the implicit to the explicit state of its form, in order directly to evaluate it; or, to discover a general method of passing from finite functions to their infinitely small elements, and reciprocally, in order to sweep the case of treating commensurable relations into the more generic one of treating incommensurable; in short, our object may be either to find the exact *discrete* numerical law which constitutes the *form* of our function, or the *continuous* numerical law which constitutes the *element-form* of the function; and thus the whole calculus does break up into a Lower and a Higher; the former having for its object that order of transformations which ends in making the function *explicit* indeed, but leaves the *law* of

the form unchanged; and the latter, that order which enables us to interchange the *form* and its *element-form*, passing in general from one law of derivation to another entirely different and more manageable.*

We thus define the Lower Calculus (*Ordinary Algebra*) as the *Calculus of Equivalent Forms*, whose object is to interchange the implicit and explicit state of a function; and the Higher Calculus as the *Calculus of Element-Forms*, whose object is to interchange the finite and infinitesimal form of a function. But this transcendental transformation may take place upon two conditions regarding the finite form: either this may be regarded as *constant*, or it may be viewed as *variable* — undergoing a perpetual and continuous change, our problem being to determine what constant form it must assume in order to satisfy certain conditions. And this last distinction points out, in the most generic way, the difference between the Differential and Integral Calculus on the one hand, and the Calculus of Variations on the other. So that we define the Differential and Integral Calculus as the *Higher Calculus of Constant Forms*, and the Calculus of Variations as the *Calculus of Variable Forms.*

The details of this scheme are recapitulated in the table annexed. I hope that it covers the ground with entire generality. That the New Geometry, the Modern Algebra, and the Quaternion, have no distinct place in it, is simply because, as it thus far seems to me, they are but modifications of the same general ideas for which the scheme already provides. Certainly the New Geometry has no logical ground distinct from that of the old—it is merely a group of new and remarkable theorems, some of which have such a sweeping generality as to become the fruitful sources of a *method* of demonstration. Likewise, the Modern Algebra is merely a modification —on a higher plane, to be sure—of the ideas constituting our ordinary algebra: it is a contrivance for the more rapid passage from the implicit to the explicit in complicated cases.

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* Thus, the ratio of the semi-circumference to the diameter being incommensurable, we cannot express the one as a finite function of the other. But the *differential* of the semi-circumference *is* a finite function of the *differential* of the diameter, and its form is the very simple one, $ds = dx \left\{ \dfrac{r^2}{2rx - x^2} \right\}^{\frac{1}{2}}$

And the Quaternion itself, so far as I can at present judge of its logical character, is also within the scope of the wide conception of Algebra as the Science of Transforming Functions: it appears to differ from the present Algebra in what its algorithm connotes—as it represents not merely magnitude (as does the present Algebra), but magnitude, direction, and position at once.

I ought not to close this article without alluding to the points which it still leaves as desiderata. It proceeds upon an assumption that our familiarity with the concept Quantity may be taken for granted; and so, also, of the notions of Space and Time. A perfect dialectic of the subject would have cleared up this obscurity, and shown us the precise logical nature of these thought-elements : this work awaits execution, unless we accept the proffered dialectic of Hegel. I am obliged to acknowledge that I have not yet been able to see that we must do so.

MATHEMATICS, THE FORM OF INTELLIGENCE IN WHICH WE SEEK TO BRING PHENOMENA UNDER THE CATEGORY OF NUMBER,

MAY BE SUBDIVIDED INTO

A. SPECULATIVE=the Science of the Functional Laws by which to convert Figure and Force into Number.

   I. THE MATHESIS = the Science which *creates* Functions, by establishing the Laws of Derivation.

      1. *Geometry* = the Science of Figure :

         A. Computative, in which the Derivation of Quantities is left to be effected by *Calculation.*

           a. Pure Geometry = the Befunctioning of *Magnitude.*

           b. Analytic Geometry = the Befunctioning of *Form.*

         B. Constructive, in which Derivation is effected by *Proportional Drawing.*

           a. Graphics = Constructive Geometry of Plane Figures; or, *Direct* Constructive Geometry.

              α. Lineal Geometry, which employs the Ruler only.

              β. Geometric Construction, which employs the Ruler and Compasses.

              γ. Mechanical Construction, which employs the Instruments generating Higher Curves.

           b. Descriptive Geometry = Constructive Geometry of Solid Figures; or, *Indirect* Constructive Geometry.

      2. *Mechanics* = the Science of Force:

         A. Statics = the Science of *Equilibrium.*

         B. Dynamics = the Science of *Motion.*

II. THE CALCULUS = the Science which *computes* Functions, either in *Form* or in *Value.*

  1. *Algebra* = the Science of the *Transformation* of Functions:

   A. Lower Calculus (*Common Algebra*) = Calculus of *Equivalent Forms,* whose object is to pass from the *Implicit* to the *Explicit* state of a Form.

   B. Higher Calculus = Calculus of *Element-Forms,* whose object is to interchange Finite and Infinitesimal Forms.

    a. Differential and Integral Calculus = Higher Calculus of *Constant* Forms.

    b. Calculus of Variations = Higher Calculus of *Variable* Forms.

  2. *Arithmetic* = the Science of the *Evaluation* of Functions :

   A. Theory of Numbers.

   B. Theory of Operations.

B. APPLIED = the Use of the Science in reducing Phenomena to Number.

  I. ASTRONOMY :

   1. *Celestial Mechanics.*

   2. *Calculus of Observations.*

  II. SURVEYING :

   1. *Geodesic.*

   2. *Topographical.*

   3. *Mensurative.*

  III. NAVIGATION :

   1. *Celesto-Spherical* (Nautical Astronomy).

   2. *Plane* (Sailing by Dead Reckoning).

  IV. PHYSICO-MATHEMATICS :

   1. *Acoustics.*

   2. *Optics.*

   3. *Thermodynamics.*

   4. *Molecular Mechanics.*

   5. *Physico-Mechanics :*

    A. Theoretical — of Rigids = Statics and Dynamics.

     — of Fluids = Hydrostatics and Hydrodynamics.

    B. Practical:

     a. Architecture, with the subsidiary art of Stereotomy.

     b. Engineering.

      $\alpha.$ Civil — Mining, Road-Building, Bridging.

      $\beta.$ Military — Fortification, Gunnery.

      $\gamma.$ Naval — Ship-modelling, Armoring.

      $\delta.$ Mechanical — Calculations for Machinery and Motors.